

Dynamics of Carroll Particles

Eric Bergshoeff¹, Joaquim Gomis² and Giorgio Longhi³

¹ *Centre for Theoretical Physics, University of Groningen,
Nijenborgh 4, 9747 AG Groningen, The Netherlands*

² *Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del
Cosmos, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain*

³ *Department of Physics and Astronomy, University of Florence, 50019 Sesto F., Firenze,
Italy; Sezione di Firenze, INFN, 50019 Sesto F., Firenze, Italy*

ABSTRACT

We investigate particles whose dynamics is invariant under the Carroll group. Although a single free such Carroll particle has no non-trivial dynamics ('the Carroll particle does not move') we show that there exists non-trivial dynamics for a set of interacting Carroll particles. Furthermore, we gauge the Carroll algebra and couple the Carroll particle to these gauge fields. It turns out that for such a coupled system even a single Carroll particle can have non-trivial dynamics.

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1 Introduction

A long time ago Bacry and Levy-Leblond [1] have classified the possible relativity groups in four dimensions. A relativity group is here defined as a possible invariance group of a 4D physical theory that contains the generators of relativity, i.e. time translations, space translations, spatial rotations and boosts. Apart from the well-known Poincaré and Galilei groups their classification also yielded the adS, dS and Newton-Hooke groups [2] and, furthermore, the less well-known Carroll groups [3]¹. They also proved how these different relativity groups are related by Inönü-Wigner contractions. In fact, all of them can be obtained by a contraction of the adS and dS groups. It turns out that there are three different types of contractions of these groups: (1) the one that takes the radius of curvature of the adS/dS spacetime to infinity; (2) the non-relativistic contraction that takes the velocity of light to infinity and (3) the Carroll contraction that takes the velocity of light to zero. In some sense the Carroll contraction is the opposite of a non-relativistic contraction and can be viewed as the ultra-relativistic limit. Such ultra-relativistic limits have been studied in, e.g., [4, 5]. Recently, it has been pointed out that there is an interesting relationship between the Carroll symmetries and the BMS algebra [6]. The BMS algebra [7] has emerged recently as the boundary symmetry group in a study of flat space holography, see for example [8, 9].

It is the purpose of this paper to study the dynamics of particles that realize the Carroll symmetries. It is not obvious that such ‘Carroll particles’ allow for any non-trivial particle dynamics. In fact, it turns out that the free Carroll particle has no dynamics at all: the free Carroll particle does not move [10, 11, 12]. In this paper we wish to investigate whether this is the generic situation or whether Carroll symmetries can allow for any non-trivial particle dynamics. We will first study the free Carroll particle. In particular, we will show that the mass-shell constraint allows for positive and negative energies and that the energy is proportional to the mass of the particle, it can be positive, negative or zero. The latter case corresponds to massless Carroll particles. The quantization of the free Carroll particle leads to a kind of ultra-local relativistic Poincaré theory. We will also study the symmetries of free Carroll particles and conclude that the Carroll symmetry is enlarged to an infinite dimensional symmetry. The particle models we consider in this paper are obtained by taking the Carroll limit of the relativistic particle. As we will see, this Carroll limit wipes out all information about the curvature of the spacetime in which the original relativistic particle was moving in.

¹In that time supersymmetry was not yet discovered.

Next, we will extend the analysis to two-particle systems in a flat spacetime and show that, in contrast to the single particle case, there is non-trivial dynamics. Moreover, we will show that the infinite-dimensional symmetry of the free Carroll particle collapses to a finite-dimensional global Carroll symmetry. We will describe one more situation in which the Carroll particle can have non-trivial dynamics. That happens when we gauge the Carroll algebra and use this as input to construct the coupling of the Carroll gauge fields to the particle. We will show that the Carroll particle in such a non-trivial background can have non-trivial dynamics.

The organization of this paper is as follows. In section 2 we introduce the free Carroll particle and identify its infinite dimensional symmetries. In section 3 we extend the analysis and consider a model of two interacting Carroll particles. In the next section we investigate the dynamics of this model and show that, unlike the free case, there is non-trivial dynamics. Subsequently, in section 5 we gauge the Carroll algebra and introduce the Carroll gauge fields. In the same section we consider the coupling of these gauge fields to the Carroll particle and again we will show that there is non-trivial dynamics. Finally, the conclusions are given in section 6.

2 The Free Carroll Particle

One way to obtain the action of the free Carroll particle is to start from the massive particle moving in an adS or dS spacetime and to take the Carroll limit². The canonical form of the action before taking the limit is given by³

$$S = \int d\tau [p \cdot \dot{x} - \frac{e}{2}(p^2 + m^2)], \quad (2.1)$$

where $p^2 = g^{\mu\nu}(x)p_\mu p_\nu$ and $g^{\mu\nu}(x)$ is the inverse metric of an adS or dS space. We will work in a basis in which the adS line element is given by

$$ds^2 = -\cosh^2 \frac{r}{R} (dX^0)^2 + \left(\frac{\sinh \frac{r}{R}}{\frac{r}{R}} \right)^2 (dX^a)^2 - \left(\left(\frac{\sinh \frac{r}{R}}{\frac{r}{R}} \right)^2 - 1 \right) (dr)^2, \quad (2.2)$$

where $r = \sqrt{X_a X^a}$. Similarly, the dS line element is given by

$$ds^2 = +\cos^2 \frac{r}{R} (dX^0)^2 + \left(\frac{\sin \frac{r}{R}}{\frac{r}{R}} \right)^2 (dX^a)^2 - \left(\left(\frac{\sin \frac{r}{R}}{\frac{r}{R}} \right)^2 - 1 \right) (dr)^2. \quad (2.3)$$

² The action for the free Carroll particle can alternatively be obtained by using the method of non-linear realizations [13] applied to the Carroll algebra [10] or by applying the method of coadjoint orbits [12]. More details about the first construction can be found in the appendix.

³ The signature of the metric is $(-, +, +, +)$.

We next consider the Carroll limit which is given by⁴⁵

$$x^0 = \frac{t}{\omega}, \quad p^0 = \omega E, \quad m = \omega M \quad (2.4)$$

with $\omega \rightarrow \infty$. It is understood that, before taking the limit, we rescale the Einbein variable like

$$e \rightarrow \frac{-e}{\omega^2} \quad (2.5)$$

for both the AdS and dS cases. The Carroll limit is in both cases the same and is given by

$$S_C = \int d\tau \left[-E\dot{t} + \dot{\vec{x}} \cdot \vec{p} - \frac{e}{2}(E^2 - M^2) \right]. \quad (2.6)$$

The canonical action (2.6) is invariant under the Carroll transformations

$$\begin{aligned} t' &= t + \vec{\beta} \cdot R\vec{x} + a_t, & \vec{x}' &= R\vec{x} + \vec{a}, \\ \vec{p}' &= R\vec{p} + \vec{\beta}E, & E' &= E. \end{aligned} \quad (2.7)$$

We observe that the Carroll limit has eliminated the R -dependence of the relativistic particle action and, consequently, any sign of the curved space time we started with. This implies that the curvature of the transverse space cannot be probed by the Carroll particle. The equations of motion corresponding to the action (2.6) are rather trivial: the free Carroll particle is at rest and does not move.

The situation is rather different if we consider instead the non-relativistic limit of a particle moving in an AdS or dS spacetime. In the AdS case, taking the non-relativistic limit leads to the harmonic oscillator with frequency $\omega = 1/R$, whereas in the dS case we obtain the inverse harmonic oscillator. To be specific, upon performing the rescaling

$$x^0 = \omega t, \quad p^0 = \frac{E}{\omega}, \quad m = \omega M \quad (2.8)$$

and taking the limit $\omega \rightarrow \infty$ in the canonical action (2.1) with the AdS line-element (2.2) we obtain⁶

⁴Note that we use a dimensionless parameter ω instead of the velocity of light. Indeed, if one considers the Carrollian counterpart of the non-relativistic limit of a string, one needs to use a dimensionless parameter [14, 15].

⁵The contractions we consider in this work correspond to the ultra-relativistic limit of a world probed by particles. There are more general contractions possible that correspond to the ultra-relativistic limit of extended objects such as strings and branes.

⁶Following [14], in order to eliminate a divergent piece we introduce the coupling to a constant electromagnetic field.

$$S_C = \int d\tau (-E\dot{t} + \dot{\vec{x}} \cdot \vec{p} - e(2ME - \vec{p}^2 - \frac{1}{R^2}\vec{x}^2)). \quad (2.9)$$

Taking the same limit in the action (2.1) with the de Sitter line element (2.3) we obtain the following action:

$$S_C = \int d\tau (-E\dot{t} + \dot{\vec{x}} \cdot \vec{p} - e(2ME - \vec{p}^2 + \frac{1}{R^2}\vec{x}^2)), \quad (2.10)$$

which is the inverse harmonic oscillator. Therefore, in contrast to the Carroll limit, the relativistic limit keeps track of the information of the curved space we started with.

Going back to the Carroll particle, the mass-shell constraint of the free Carroll particle is given by

$$\phi = E^2 - M^2 = 0, \quad (2.11)$$

which is solved by $E = \pm M$. Note that the mass-shell constraints do not depend on the spatial momenta. A difference with respect to the non-relativistic case is that here we can have particles with negative energy and, furthermore, the massless limit $M \rightarrow 0$ is well-defined. The action of a massless Carroll particle becomes

$$S_C = \int d\tau (-E\dot{t} + \dot{\vec{x}} \cdot \vec{p} - \frac{e}{2}E^2). \quad (2.12)$$

If we quantize the model and impose the mass-shell constraint on the physical states we end up with a kind of ultra-local relativistic Poincaré theory with wave equation

$$\left(-\frac{d^2}{dt^2} - M^2\right)\phi(t, \vec{x}) = 0. \quad (2.13)$$

Solving for the mass-shell constraints (2.11) we can write the action in the equivalent form

$$S_C = \int d\tau [\mp M\dot{t} + \dot{\vec{x}} \cdot \vec{p}]. \quad (2.14)$$

In this case the transformation of the momenta becomes $\vec{p}' = R\vec{p} \pm \vec{\beta}M$.

The generators of the Carroll algebra are given by

$$H = E, \quad \vec{P} = \vec{p}, \quad \vec{G} = E\vec{x}, \quad \vec{J} = \vec{x} \times \vec{p}, \quad (2.15)$$

which is to be supplemented by the mass-shell condition $E = \pm M$. In the massless case the generators are given by

$$H = 0, \quad \vec{P} = \vec{p} = p\vec{u}, \quad \vec{G} = 0\vec{x}, \quad \vec{J} = \vec{x} \times \vec{p}, \quad (2.16)$$

where \vec{u} is a unit vector, i.e. a general element of S^2 [16].

We will now dedicate a separate subsection to a discussion of the symmetries of the free Carroll particle.

2.1 Infinite-dimensional Symmetries

The basic Poisson brackets of the canonical variables occurring in the action (2.6) are given by

$$\{E, t\} = 1, \quad \{e, \pi\} = 1, \quad \{x_i^m, p_j^n\} = \delta_{m,n} \delta^{i,j}. \quad (2.17)$$

This leads to the following equations of motion for these variables:

$$\dot{t} = -eE, \quad \dot{\vec{x}} = 0, \quad \dot{e} = \lambda(\tau), \quad \dot{E} = 0, \quad \dot{\vec{p}} = 0, \quad \dot{\pi} = -1/2(E^2 - M^2), \quad (2.18)$$

where $\lambda(\tau)$ is an arbitrary function and π is the momenta associated to the Einbein variable e which is constrained by $\pi = 0$.

Consider now the following generator of canonical transformations

$$G = -E\xi^0(\vec{x}, t) + p_i \xi^i(\vec{x}, t) + \gamma(\vec{x}, t)\pi, \quad (2.19)$$

with parameters $\xi^0(\vec{x}, t)$, $\xi^i(\vec{x}, t)$ and $\gamma(\vec{x}, t)$. The transformations generated by this generator are given by

$$\begin{aligned} \delta t &= \xi^0(\vec{x}, t), \quad \delta x^i = \xi^i(\vec{x}, t), \quad \delta e = \gamma(\vec{x}, t), \\ \delta E &= -\partial_t \xi^0(\vec{x}, t)E + \partial_t \xi^i(\vec{x}, t)p_i + \partial_t \gamma(\vec{x}, t)\pi, \\ \delta p_i &= \partial_i \xi^0(\vec{x}, t)E - \partial_i \xi^j(\vec{x}, t)p_j - \partial_i \gamma(\vec{x}, t)\pi. \end{aligned} \quad (2.20)$$

These transformations are symmetries of the free Carroll particle, provided that G is a constant of motion, i.e., $\partial_\tau G = 0$. This leads to the following restriction on the parameters:

$$\begin{aligned} 0 &= -E(\dot{t}\partial_t \xi^0(\vec{x}, t) + \dot{x}_j \partial^j \xi^0(\vec{x}, t)) + p_i(\dot{t}\partial_t \xi^i(\vec{x}, t) + \dot{x}_j \partial^j \xi^i(\vec{x}, t)) + \dot{\pi}\gamma(\vec{x}, t) \\ &= -eE^2 \partial_t \xi^0(\vec{x}, t) - eE p_i \partial_t \xi^i(\vec{x}, t) - \frac{1}{2}\gamma(\vec{x}, t)(E^2 - M^2). \end{aligned} \quad (2.21)$$

From this equation we deduce the following Killing equations corresponding to the free Carroll particle:

$$\partial_t \xi^0(\vec{x}, t) = 0, \quad \partial_t \xi^i(\vec{x}, t) = 0, \quad \gamma(\vec{x}, t) = 0. \quad (2.22)$$

The solutions of these Killing equations are

$$\xi^0 = \xi^0(\vec{x}), \quad \xi^i = \xi^i(\vec{x}) \quad (2.23)$$

and, hence, the generator G is given by

$$G = -E\xi^0(\vec{x}) + p_i \xi^i(\vec{x}). \quad (2.24)$$

We thus conclude that the free Carroll particle has an infinite dimensional symmetry. The Carroll transformations (2.7) are obtained by keeping the first term in the powers series expansion of the parameters $\xi^0(\vec{x})$ and $\xi^i(\vec{x})$ in terms of \vec{x} . Adding some curvature structure by hand to the transverse space will eliminate the "transverse spatial" transformations as in [12, 6].

In the special case of a massless Carroll particle the isometries should be given by the most general conformal Carroll group. The Killing equations in this case become

$$\partial_t \xi^0(t, \vec{x}) - \frac{\gamma}{2e} = 0, \quad \partial_t \xi^i(t, \vec{x}) = 0 \quad (2.25)$$

for arbitrary parameter $\gamma(\vec{x}, t)$. This leads to the following generator of conformal Killing transformations

$$G = -E\xi^0(t, \vec{x}) + p_i \xi^i(\vec{x}) + 2\pi_e \partial_t \xi^0(t, \vec{x}), \quad (2.26)$$

which again generates an infinite dimensional symmetry. These transformations include scale transformations of the time and space coordinates. If we put more structure in the transverse space these transformations will have restrictions and we could obtain the Carroll transformations of [16, 6].

This concludes our discussion of the free Carroll particle, its dynamics and its symmetries.

3 A Model of Two Interacting Carroll Particles

In this section we will extend the analysis of the previous section and consider a model of two Carroll particles interacting through a potential V that depends on the relative variables of the particles. In order to construct the model we will first consider in subsection 3.1 a relativistic model of two interacting particles. Next, in subsection 3.2, we will consider the Carroll limit of this relativistic model. We will show that, unlike the single particle case, there is non-trivial dynamics in the two body system. It is sufficient to consider the two relativistic particles in flat space-time since, as we have seen in the single particle case, the Carroll limit eliminates any reference to the curvature of the spacetime we start with. Finally, in subsection 3.3 we will investigate the symmetries of the interacting Carroll model.

3.1 A Relativistic Two Particle Model

We consider the interacting two relativistic particle model of [17, 18, 19]. This model can be defined on a phase space, where the coordinates and momenta are x_i^μ, p_i^μ , ($i = 1, 2; \mu =$

0, 1, 2, 3). The basic Poisson brackets are given by ⁷

$$\{x_i^\mu, p_j^\nu\} = \delta_{i,j} \eta_{\mu\nu}. \quad (3.1)$$

The model is defined in terms of two constraints ϕ_1 and ϕ_2 given by

$$\phi_1 = p_1^2 + m_1^2 + V, \quad \phi_2 = p_2^2 + m_2^2 + V.$$

We assume that the potential V has the most general dependence on the variables allowed by the requirement that the following first class condition holds:

$$\{\phi_1, \phi_2\} = 0. \quad (3.2)$$

The constraints ϕ_1 and ϕ_2 are just a modification of the two mass shell constraints of two free particles through the potential term. This potential term breaks the two Poincaré invariances of the two free particles to a diagonal Poincaré invariance ⁸.

As shown, for instance in [23], the potential V can have a dependence on the set of scalars that are formed from the variables p_1^μ and p_2^μ , and the relative coordinate $r^\mu = x_1^\mu - x_2^\mu$, transverse to the total momentum $P^\mu = p_1^\mu + p_2^\mu$. This can be seen as follows. The first class condition (3.2) can be written as

$$p_1^\mu \frac{\partial V}{\partial x_1^\mu} - p_2^\mu \frac{\partial V}{\partial x_2^\mu} = 0. \quad (3.3)$$

We must add to this the requirement of translation invariance, that is

$$\frac{\partial V}{\partial x_1^\mu} = -\frac{\partial V}{\partial x_2^\mu}, \quad (3.4)$$

so that the first class condition becomes

$$P^\mu \frac{\partial V}{\partial r^\mu} = 0. \quad (3.5)$$

This shows that indeed V must depend on the scalars formed from p_1, p_2 and the part of r that is transverse to the total momentum P^μ .

The allowed scalars are given by

$$\begin{aligned} s_1 &= -r_\perp^2, & s_2 &= (r_\perp, p_1), & s_3 &= (r_\perp, p_2) = -s_2, \\ s_4 &= -p_1^2, & s_5 &= -p_2^2, & s_6 &= -(p_1, p_2). \end{aligned} \quad (3.6)$$

⁷Remember that the signature of η is $(-, +, +, +)$.

⁸There is also a model where only one combination of the mass shell constraints is first class and there is a further transversality constraint that is second class [20, 21, 22]. This model has not a well defined Carroll limit and will not be considered here.

In these equations the transverse relative variable r_\perp is defined by

$$r_\perp^\mu = r^\mu - \frac{(P \cdot r)}{P^2} P^\mu, \quad (3.7)$$

where $\vec{r} = \vec{x}_1 - \vec{x}_2$ and $\vec{P} = \vec{p}_1 + \vec{p}_2$.

This finishes our discussion of the relativistic two body model. In the next subsection we will take the Carroll limit of this model.

3.2 The Carroll Limit of the Two Body Relativistic Model

The Carroll limit of the relativistic two body model defined in the previous subsection is defined by rescaling the canonical variables with a parameter ω as follows:

$$\begin{aligned} p_1^0 &= \omega E_1, & p_2^0 &= \omega E_2, & x_1^0 &= \frac{1}{\omega} t_1, \\ x_2^0 &= \frac{1}{\omega} t_2, & m_1 &= \omega M_1, & m_2 &= \omega M_2 \end{aligned}$$

and taking the limit $\omega \rightarrow \infty$.

The Carroll symmetries of the model we are looking for should be given by

$$\begin{aligned} t'_i &= t_i + \vec{\beta} \cdot \vec{x}_i + a_t, & \vec{x}'_i &= \vec{x}_i + \vec{a}, \\ E'_i &= E_i, & \vec{p}'_i &= \vec{p}_i + \vec{\beta} E_i, \quad (i = 1, 2). \end{aligned} \quad (3.8)$$

These transformations are the Carroll limit of the diagonal Poincaré symmetries of the two particles.

In order to derive the dependence of the Carroll potential on the coordinates we should isolate in the asymptotic expansion of the scalars those terms that scale like ω^2 :

$$\begin{aligned} s_4 &= \omega^2 E_1^2 - (\vec{p}_1)^2 + O(\omega^{-2}), & s_5 &= \omega^2 E_2^2 - (\vec{p}_2)^2 + O(\omega^{-2}), \\ s_6 &= \omega^2 E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 + O(\omega^{-2}), & s_4 s_5 - s_6^2 &= -\omega^2 (E_1 \vec{p}_2 - E_2 \vec{p}_1)^2 + \dots \end{aligned} \quad (3.9)$$

Note that in the last equation the square of the Carroll boost invariant vector

$$\vec{q} \equiv E_1 \vec{p}_2 - E_2 \vec{p}_1 \quad (3.10)$$

occurs.

We conclude that the potential could depend on q^2, E_1 and E_2 , i.e., $V = V(q^2, E_1, E_2)$. Furthermore, since $\vec{r} = \vec{x}_1 - \vec{x}_2$ is also a Carroll boost invariant vector we could have the more general potential $V = V(r^2, q^2, \vec{r} \cdot \vec{q}, E_1, E_2)$. Therefore, the two first class Carroll

invariant mass shell constraints ϕ_i ($i = 1, 2$) which satisfy the constraint $\{\phi_1, \phi_2\} = 0$ and define the model are given by

$$\phi_i = E_i^2 - M_i^2 - V(\vec{r}^2, \vec{q}^2, \vec{r} \cdot \vec{q}, E_1, E_2). \quad (3.11)$$

Note that, due to the dependence of the potential on $(\vec{r})^2$ the two particle model is sensitive to the curvature of the transverse space. We consider here a flat transverse space. This leads to the following canonical action for the model

$$S = \int d\tau \mathcal{L} = \int d\tau \left[E_1 \dot{t}_1 + E_2 \dot{t}_2 - \dot{\vec{x}}_1 \cdot \vec{p}_1 - \dot{\vec{x}}_2 \cdot \vec{p}_2 + \lambda_1 \phi_1 + \lambda_2 \phi_2 \right]. \quad (3.12)$$

We observe that the infinite-dimensional symmetries of the single particle are absent in the two-particle model due to the non-invariance of the relative coordinates \vec{r} and \vec{q} .

Having defined the interacting Carroll model, we are going to investigate its dynamics in the next section.

4 The Dynamics of Two Carroll Particles

The equations of motion derived from the canonical action (3.12) are given by

$$\begin{aligned} \dot{\vec{x}}_1 &= \{\vec{x}_1, -V\}(e_1 + e_2), & \dot{\vec{x}}_2 &= \{\vec{x}_2, -V\}(e_1 + e_2), \\ \dot{t}_1 &= -2E_1 e_1 + \{t_1, -V\}(e_1 + e_2), & \dot{t}_2 &= -2E_2 e_2 + \{t_2, -V\}(e_1 + e_2), \\ \dot{\vec{p}}_1 &= \{\vec{p}_1, -V\}(e_1 + e_2), & \dot{\vec{p}}_2 &= \{\vec{p}_2, -V\}(e_1 + e_2), \\ \dot{E}_1 &= \{E_1, -V\}(e_1 + e_2), & \dot{E}_2 &= \{E_2, -V\}(e_1 + e_2). \end{aligned} \quad (4.1)$$

We wish to investigate whether in general two interacting Carroll particles can move, i.e. can have non-trivial dynamics. In order to find out we choose as an example a simple potential that only depends on the “relative momenta” $\vec{q} = E_1 \vec{p}_2 - E_2 \vec{p}_1$, i.e., we consider a potential of the form $V = V(\vec{q}^2)$. In this special case the equations of motion of \vec{x}_i read

$$\dot{\vec{x}}_1 = -\frac{4}{D} V' E_2 \vec{q} (E_2 \dot{t}_1 + E_1 \dot{t}_2), \quad \dot{\vec{x}}_2 = +\frac{4}{D} V' E_1 \vec{q} (E_2 \dot{t}_1 + E_1 \dot{t}_2), \quad (4.2)$$

where we have used the equations of motion (4.1) to write the Einbein variables in terms of \dot{t}_i and a new variable D which is defined by

$$D = 4[E_1 E_2 + E_1 V'(\vec{p}_1 \cdot \vec{q}) - E_2 V'(\vec{p}_2 \cdot \vec{q})]. \quad (4.3)$$

We fix the two gauge symmetries generated by the two first class constraints ϕ_1 and ϕ_2 by imposing the gauge-fixing conditions $t_1 = t_2 = t$. Substituting these conditions back into the equations of motion we obtain

$$\frac{d\vec{x}_1}{dt} = -E_2 \frac{4(E_1 + E_2)}{D} V' \vec{q}, \quad \frac{d\vec{x}_2}{dt} = +E_1 \frac{4(E_1 + E_2)}{D} V' \vec{q}. \quad (4.4)$$

From this we derive that

$$\frac{d}{dt}(E_1 \vec{x}_1 + E_2 \vec{x}_2) = 0. \quad (4.5)$$

In this equation we still need to express E_1 and E_2 using the mass shells constraints (3.11). For our choice of potential we can express E_1 and E_2 in terms of the momenta p_1, p_2 and the masses M_1, M_2 . We have four sheets of solutions. Introducing a small parameter α in the potential we can write

$$E_i = M_i + O_i(\alpha). \quad (4.6)$$

We find that to lowest order in α the velocity of the center of mass is conserved, i.e.,

$$M_1 \frac{d\vec{x}_1}{dt} + M_2 \frac{d\vec{x}_2}{dt} = \text{constant}. \quad (4.7)$$

This implies non-trivial dynamics for the separate particles.

Our final conclusion is that, in contrast to a single Carroll particle, interacting Carroll particles can have non-trivial dynamics!

5 Coupling the Carroll Particle to Gauge Fields

In this section we will consider the gauging of the Carroll algebra and consider the coupling of the Carroll particle to the gauge fields corresponding to the Carroll algebra. In the first subsection we will introduce the Carroll algebra and compare it with the well-known Galilei algebra. It is known that a gauging of the Galilei algebra, or more precisely its centrally extended version, the Bargmann algebra, leads to a description of Newton-Cartan gravity. In the second subsection we will investigate what happens when one applies the same gauging procedure to the (non-centrally extended) Carroll algebra. Finally, in the last subsection we will consider the coupling of the Carroll particle to the gauge fields of the Carroll algebra.

5.1 Comparing the Galilei and Carroll Algebras

The Galilei and Carroll algebras can be viewed as different contractions of the Poincaré algebra. We therefore start by considering the Poincaré algebra in D spacetime dimensions

$$[M_{BC}, P_A] = -2\eta_{A[B} P_{C]}, \quad [M_{CD}, M_{EF}] = 4\eta_{[C[E} M_{F]D]}, \quad (5.1)$$

where the indices $A, B, \dots = 0, 1, \dots, D-1$ are flat Lorentz indices. Writing $A = (0, a)$, $a = 1, 2, \dots, D-1$, the usual non-relativistic limit of the Poincaré algebra is defined by means of the following contraction:

$$\text{Galilei contraction:} \quad P_0 = \frac{1}{\omega} H, \quad M_{a0} = \omega G_a, \quad \omega \rightarrow \infty, \quad (5.2)$$

which leads to the D -dimensional Galilei algebra:

Galilei Algebra

$$\begin{aligned} [J_{ab}, P_c] &= -2\delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= -2\delta_{c[a} G_{b]}, \\ [J_{cd}, J_{ef}] &= 4\eta_{[c[e} J_{f]d]}, & [G_a, H] &= -P_a. \end{aligned} \quad (5.3)$$

We have renamed $M_{ab} = J_{ab}$. Here (H, P_a, J_{ab}, G_a) are the generators of time translations, space translations, boosts and spatial rotations, respectively.

The Galilei transformations corresponding to the algebra (5.3) acting on spacetime coordinates $x^\mu = (t, x^i)$, $i = 1, 2, \dots, D-1$, are given by

$$\delta t = -\zeta, \quad \delta x^i = \lambda^i_j x^j - v^i t - a^i. \quad (5.4)$$

Here $(\zeta, \lambda^i_j, v^i, a^i)$ parametrize a (constant) time translation, space translation, spatial rotation and boost transformation, respectively. A special feature of the Galilei algebra is that it admits a central extension. The centrally extended Galilei algebra contains an additional central charge generator Z and is called the Bargmann algebra. The commutators of the Bargmann algebra are given by those of the Galilei algebra, see eq. (5.3), together with the following commutator containing Z :

$$[G_a, P_b] = -\delta_{ab} Z. \quad (5.5)$$

It turns out that this central extension is indispensable in order to show that Newton-Cartan gravity follows from the gauging of an algebra.

There exists another less well-known contraction of the Poincaré algebra which corresponds to taking the ultra-relativistic limit. This so-called Carroll contraction is given by [3]:

$$\text{Carroll contraction:} \quad P_0 = \omega H, \quad M_{a0} = \omega G_a, \quad \omega \rightarrow \infty. \quad (5.6)$$

This contraction leads to the D -dimensional Carroll algebra [3]:

Carroll Algebra

| symmetry | generators | gauge field | parameters | curvatures |
|-----------------------|------------|---------------------|-----------------------|------------------------|
| temporal translations | H | τ_μ | $\zeta(x^\nu)$ | $R_{\mu\nu}(H)$ |
| spatial translations | P_a | $e_\mu{}^a$ | $\zeta^a(x^\nu)$ | $R_{\mu\nu}{}^a(P)$ |
| boosts | G_a | $\omega_\mu{}^a$ | $\lambda^a(x^\nu)$ | $R_{\mu\nu}{}^a(G)$ |
| spatial rotations | J_{ab} | $\omega_\mu{}^{ab}$ | $\lambda^{ab}(x^\nu)$ | $R_{\mu\nu}{}^{ab}(J)$ |

TABLE 1

This table indicates the generators of the Carroll algebra and the gauge fields, local parameters and curvatures that are associated to each of these generators.

$$\begin{aligned}
[J_{ab}, P_c] &= -2\delta_{c[a}P_{b]} , & [J_{ab}, G_c] &= -2\delta_{c[a}G_{b]} , \\
[J_{cd}, J_{ef}] &= 4\eta_{[c[e}J_{f]d]} , & [G_a, P_b] &= -\delta_{ab}H .
\end{aligned} \tag{5.7}$$

In contrast to the Galilei algebra, the Carroll algebra does not allow for a central extension⁹. The Carroll transformations corresponding to the algebra (5.7) acting on spacetime coordinates $x^\mu = (t, x^i)$, $i = 1, 2, \dots, D-1$, are given by

$$\delta t = -\zeta - v^i x^i , \quad \delta x^i = \lambda^i{}_j x^j - a^i . \tag{5.8}$$

Here, like in the Galilei case, $(\zeta, \lambda^i{}_j, v^i, a^i)$ parametrize a (constant) time translation, space translation, spatial rotation and boost transformation, respectively.

Below we derive the gauge transformations of the Carroll gauge fields by gauging the Carroll algebra thereby stressing the common features as well as the differences with the gauging of the centrally extended Galilei algebra, i.e. the Bargmann algebra.

5.2 Gauging the Carroll Algebra

We consider the gauging of the D -dimensional Carroll algebra (5.7) following the same gauging procedure that in the case of the Bargmann algebra leads to Newton-Cartan gravity [24, 25] and see how far we can get. As a first step we introduce for each generator of the Carroll algebra a gauge field, a local parameter parametrizing the corresponding symmetry and the gauge-covariant curvatures, see Table 1.

⁹An exception is the 3D Carroll algebra which does allow a central extension of the form $[G_a, G_b] = \epsilon_{ab}\tilde{Z}$. Since we wish to consider the generic situation, valid for D dimensions, we will not consider this central extension any further.

According to the Carroll algebra (5.7) the gauge fields transform as follows:¹⁰

$$\begin{aligned}
\delta\tau_\mu &= \lambda^a e_{\mu a} + \partial_\mu \zeta - \zeta^a \omega_\mu{}^a, \\
\delta e_\mu{}^a &= (D_\mu \zeta)^a + \lambda^a{}_b e_\mu{}^b, \\
\delta\omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab}, \\
\delta\omega_\mu{}^a &= (D_\mu \lambda)^a + \lambda^a{}_b \omega_\mu{}^b,
\end{aligned} \tag{5.9}$$

where D_μ is the covariant derivative with respect to spatial rotations. The following curvatures transform covariantly under these transformations:

$$\begin{aligned}
R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a}, \\
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]b}, \\
R_{\mu\nu}{}^a(G) &= 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}, \\
R_{\mu\nu}{}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab}.
\end{aligned} \tag{5.10}$$

Our first task is now to impose conventional constraints on the curvatures, like one does when gauging the Bargmann algebra [26]. To be precise, we impose constraints on the Carroll curvatures (5.10) such that the temporal and spatial translations, with parameters ζ and ζ^a , get equivalent to the general coordinate transformations, with parameters ξ^μ , modulo boosts, with parameters $\lambda^a = \xi^\mu \omega_\mu{}^a$ and spatial rotations, with parameters $\lambda^{ab} = \xi^\mu \omega_\mu{}^{ab}$. For this, we need the following two identities for those gauge fields that transform under H and/or P_a -transformations.:

$$\delta_{\text{g.c.t.}}(\xi^\nu)\tau_\mu = \left[\delta_H(\xi^\lambda \tau_\lambda) + \delta_P(\xi^\lambda e_\lambda{}^a) + \delta_G(\xi^\lambda \omega_\lambda{}^a) \right] \tau_\mu + \xi^\lambda R_{\lambda\mu}(H), \tag{5.11}$$

$$\delta_{\text{g.c.t.}}(\xi^\nu)e_\mu{}^a = \left[\delta_P(\xi^\lambda e_\lambda{}^a) + \delta_J(\xi^\lambda \omega_\lambda{}^{ab}) \right] e_\mu{}^a + \xi^\lambda R_{\lambda\mu}{}^a(P). \tag{5.12}$$

These identities show that, in order to equate a general coordinate transformation to an H - and P_a -transformation, modulo a boost and/or a spatial rotation, we need to impose the following set of conventional constraints:

$$R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = 0. \tag{5.13}$$

We furthermore deduce that the relation between the different parameters is given by

$$\zeta = \xi^\mu \tau_\mu, \quad \zeta^a = \xi^\mu e_\mu{}^a. \tag{5.14}$$

¹⁰All parameters depend on the coordinates x^μ , even when not explicitly indicated.

Introducing the projective inverses τ^μ and e^μ_a of τ_μ and e_μ^a , respectively, as follows:

$$\begin{aligned} e_\mu^a e^\mu_b &= \delta_b^a, & \tau^\mu \tau_\mu &= 1, \\ \tau^\mu e_\mu^a &= 0, & \tau_\mu e^\mu_a &= 0, \\ e_\mu^a e^\nu_a &= \delta_\mu^\nu - \tau_\mu \tau^\nu. \end{aligned} \tag{5.15}$$

we derive that the inverse relation between ζ, ζ^a and ξ^μ is given by

$$\xi^\mu = \tau^\mu \zeta + e^\mu_a \zeta^a. \tag{5.16}$$

The gauge fields τ_μ and e_μ^a can now be interpreted as the temporal and spatial Vielbeine. The transformations of these Vielbeine together with their projective inverse fields under boosts and spatial rotations are given by

$$\delta \tau_\mu = \lambda^a e_{\mu a}, \quad \delta e_\mu^a = \lambda^a_b e_\mu^b, \tag{5.17}$$

$$\delta \tau^\mu = 0, \quad \delta e^\mu_a = \lambda_a^b e^\mu_b - \tau^\mu \lambda_a, \tag{5.18}$$

while under general coordinate transformations they transform as covariant (τ_μ and e_μ^a) and contra-variant (τ^μ and e^μ_a) vectors. From now on we will work solely with the general coordinate transformations and not consider the temporal and spatial translations anymore.

We observe that the projective invertability relations (5.15) are invariant under the non-trivial boost transformations of τ_μ and e^μ_a . Like in the Bargmann case, this corresponds to the ambiguity of defining the inverse of a singular matrix. Therefore, in the Carroll case the only fields that are un-ambiguously defined are

$$\text{Carroll:} \quad \{\tau^\mu, e_\mu^a\} \tag{5.19}$$

These Carroll fields are invariant under boosts and transform in the standard way under spatial rotations and general coordinate transformations. Note that there is no central charge gauge field.

Unlike in the Bargmann case, the conventional constraints (5.13) are not sufficient to solve for the boost gauge fields ω_μ^a and the gauge field of spatial rotations ω_μ^{ab} . The reason for this difference is that the Bargmann algebra leads to an additional central charge gauge field whose curvature may be set to zero. This particular conventional constraint plays a crucial role in solving for the spin-connection fields in the Bargmann case. In the Carroll case, the boost gauge field ω_μ^a only occurs in the first constraint in (5.13) and this constraint is invariant under the following shift symmetries:

$$\omega_\mu^a \rightarrow \omega_\mu^a + e_{\mu b} X^{(ab)}, \tag{5.20}$$

with $X^{(ab)}$ an arbitrary symmetric tensor. This shows that $\omega_\mu{}^a$ can only be solved modulo this ambiguity.

Another difference with the Galilei case is that a foliation-defining constraint, like the constraint $\partial_{[\mu}\tau_{\nu]}G = 0$ that we have in the Bargmann case, is absent. The reason for this is that in the Carroll case such a constraint is not invariant under boost transformations. Instead, it is the inverse temporal Vielbein $(\tau^\mu)_C$ that is invariant under boost transformations and hence has an invariant meaning. This is in line with the duality relation between the Galilei and Carroll cases discussed in [27, 12]

$$(\tau_\mu dx^\mu)_G \longleftrightarrow (\tau^\mu \frac{\partial}{\partial x^\mu})_C. \quad (5.21)$$

5.3 Coupling the Carroll Particle to Gauge Fields

Having established in the previous subsection the transformation rules of the Carroll gauge fields, let us now couple a Carroll particle to these gauge fields. Since τ^μ is invariant under boosts it is natural to use this gauge field when coupling to the Carroll particle. The covariantization of the action (2.6) which is invariant under the sigma model symmetries is

$$S = \int d\tau \left[p_\mu \dot{x}^\mu - \frac{e}{2} \left(\tau^\mu(t, \vec{x}) \tau^\nu(t, \vec{x}) p_\mu p_\nu - M^2 \right) \right], \quad (5.22)$$

where $p_0 = -E$. The equations of motion that follow from this action are given by

$$\dot{x}^\mu = e \tau^\mu \tau^\nu p_\nu, \quad \dot{p}_\mu = -e (\partial_\mu \tau^\rho) \tau^\sigma p_\sigma p_\rho. \quad (5.23)$$

From these equations we deduce that the single Carroll particle in a non-trivial background specified by the inverse Vielbein τ^μ has non-trivial dynamics.

It is not difficult to compute the Killing equations to find the Noether symmetries of the action (5.22). We take as the generator of these transformations

$$G = \xi^\mu(t, \vec{x}) p_\mu. \quad (5.24)$$

For a flat transverse space, the condition of being a symmetry, $\dot{G} = 0$, implies

$$\mathcal{L}_\xi \tau^\mu = 0. \quad (5.25)$$

If curvature is turned on by hand this condition should be supplemented with the requirement that $\mathcal{L}_\xi g_{ij} = 0$ [12].

6 Discussion

The particle models we have considered in this paper are obtained by taking the Carroll limit of the relativistic particle. The resulting Carroll particle is by construction invariant under the Carroll symmetries. A particular feature of the Carroll limit is that it wipes out all information about the curvature of the spacetime in which the original relativistic particle was moving in. The free Carroll particle has several noteworthy features. First of all, the free particle action is in fact invariant under infinite-dimensional symmetries, see eq. (2.23). Secondly, the mass-shell constraint allows for positive as well as negative energy solutions, like in the relativistic case. There exists also the limit to a massless Carroll particle.

Concerning the dynamics, the free Carroll particle has rather uninteresting dynamics: it cannot move. In this paper we investigated two situations in which this is no longer the case. We first showed that for a set of interacting Carroll particles only the center of mass cannot move but that the separate particles can have non-trivial dynamics, see eq. (4.7). Next, we introduced non-trivial background fields by gauging the Carroll algebra and coupled these background fields to a single Carroll particle. We showed that even a single Carroll particle, due to the non-trivial background fields, can move, see eq. (5.23).

When gauging the Carroll algebra, we were not able to find a set of conventional constraints that enables one to solve for the connection fields corresponding to spatial rotations and boosts. This is in contrast to what happens when gauging the Bargmann algebra where the connection fields can be solved and where the gauging procedure leads to a description of Newton-Cartan gravity. A crucial role is here played by the central charge generator which can be added to the Galilei algebra but not to the Carroll algebra. It would be interesting to see in which sense a definition of ‘Carroll gravity’ can be given consistent with the duality between Galilei and Carroll gravity proposed in [27, 12].

One direction where Carroll symmetries and the non-trivial particle dynamics could have applications is in studies of the gauge/gravity duality where BMS symmetries, which are closely related to Carroll symmetries [6, 12], emerge at the boundary. It would be interesting to investigate these relationships further and, in particular, to consider generalizations of the Carroll contraction that correspond to the ultra-relativistic limit of a world probed by extended objects such as strings and branes.

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A An Alternative Derivation of the Carroll Particle Action

In this appendix we give an alternative derivation of the Carroll particle action (2.14) using the non-linear realization method [13]. Our starting point is the D -dimensional Carroll algebra given in (5.7). We consider the coset $\frac{\text{Carroll}}{\text{Rotations}}$ [10], that we locally parametrize as

$$g = e^{-tH} e^{\vec{x}\vec{P}} e^{\vec{v}\vec{G}}, \quad (\text{A.1})$$

where t, \vec{x} are the Goldstone bosons associated to space-time translations and \vec{v} are the Goldstone bosons associated to the broken boosts.

The Maurer-Cartan form Ω is given by

$$\Omega = g^{-1}dg = HL^H + L^{\vec{P}}\vec{P} + L^{\vec{G}}\vec{G} \quad (\text{A.2})$$

$$= (-dt + \vec{v}d\vec{x})H + d\vec{x}\vec{P} + d\vec{v}\vec{G}. \quad (\text{A.3})$$

The action for the particle with lowest order in derivatives is obtained by taking the pullback of the rotation invariant form L^H to the world-line of the particle:

$$S = M \int d\tau (L^H)^* = \int d\tau (-M\dot{t} + M\vec{v}\dot{\vec{x}}). \quad (\text{A.4})$$

The momentum of the Carroll particle is therefore $\vec{p} = M\vec{v}$. The Goldstone bosons of the broken boosts are always related to the momentum of the particle. The action obtained from the non-linear realization in general is a phase space action, see for example [28].

Finally, the transformations that leave invariant the Maurer-Cartan form are the Carroll symmetries

$$\begin{aligned} t' &= t + \vec{\beta} \cdot \vec{x} + a_t, & \vec{x}' &= \vec{x} + \vec{a}, \\ \vec{v}' &= \vec{v} + \vec{\beta}. \end{aligned} \quad (\text{A.5})$$

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